Chapter 9. Cloud Equilibrium and Stability

Notes:

- *Most of the material presented in this chapter is taken from Stahler and Palla (2004), Chap. 9.*

9.1 Isothermal Spheres

As we endeavor to determine the conditions that are conducive to gravitational collapse, we first concentrate on the useful idealization of isothermal spheres. We therefore do away with any temperature variations across our idealized cloud, as well as other quantities such as magnetic fields. With these assumptions the equation of motion under the assumption of (hydrostatic) equilibrium is

$$-\frac{1}{\rho}\nabla p - \nabla\Phi_{\mathrm{g}} = 0, \tag{9.1}$$

where $\rho, p,$ and Φ_{g} are the mass density of the gas, its pressure, and the gravitational potential, respectively. For an isothermal gas the equation of state is given by

$$\begin{aligned} p &= nk_{\mathrm{B}}T \\ &= \rho\frac{\mathcal{R}}{\mu}T \\ &= \rho a_T^2, \end{aligned} \tag{9.2}$$

where $a_T = (\mathcal{R}T/\mu)^{1/2}$ is the isothermal sound speed (a constant in this case). We need one more equation to complete this system of unknowns, it is provided by Poisson's equation for the gravitational potential

$$\nabla^2\Phi_{\mathrm{g}} = 4\pi G\rho, \tag{9.3}$$

with G is the universal gravitational constant. Equations (9.1) to (9.3) can be readily solved by first taking the gradient of the pressure

$$\nabla p = a_T^2\nabla\rho, \tag{9.4}$$

which upon division by ρ transforms to

$$\frac{1}{\rho}\nabla p = a_T^2\nabla\left[\ln(\rho)\right]. \tag{9.5}$$

Inserting this result in equation (9.1) yields

$$\nabla\left[\ln(\rho)+\frac{\Phi_{\mathrm{g}}}{a_T^2}\right]=0. \tag{9.6}$$

Concentrating on systems exhibiting spherical symmetry we find that

$$\frac{d}{dr}\left[\ln(\rho)+\frac{\Phi_{\mathrm{g}}}{a_T^2}\right]=0, \tag{9.7}$$

which implies that

$$\rho(r)=\rho_{\mathrm{c}}e^{-\Phi_{\mathrm{g}}/a_T^2}, \tag{9.8}$$

with ρ_{c} the density at the origin (i.e., $r=0$), where we also set $\Phi_{\mathrm{g}}(0)=0$. Poisson's equation can now be expressed solely as a function the potential by inserting equation (9.8) into equation (9.3)

$$\begin{aligned}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi_{\mathrm{g}}}{dr}\right)&=4\pi G\rho\\&=4\pi\rho_{\mathrm{c}}e^{-\Phi_{\mathrm{g}}/a_T^2}.\end{aligned} \tag{9.9}$$

We now define the dimensionless constants

$$\begin{aligned}\psi&\equiv\frac{\Phi_{\mathrm{g}}}{a_T^2}\\ \xi&\equiv\left(\frac{4\pi G\rho_{\mathrm{c}}}{a_T^2}\right)^{1/2}r,\end{aligned} \tag{9.10}$$

and transform equation (9.9) into the so-called **isothermal Lane-Emden equation**

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi}{d\xi}\right)=e^{-\psi}. \tag{9.11}$$

This is a second order differential equation in $\psi(\xi)$, and we therefore need two initial (or boundary) conditions. We already know that $\psi(0)=0$. Moreover, since we also know that the gravitational potential is constant at the interior of a spherical shell and that a sphere can be modeled as a series of such embedded shells, then we also have that $d\psi/d\xi\big|_{\xi=0}=0$. That is, the gravitational force also vanishes at the centre of the sphere. Numerical solutions for $\psi(\xi)$ from equation (9.11) and subsequently the mass density from equation (9.8) are shown Figure 9.1.

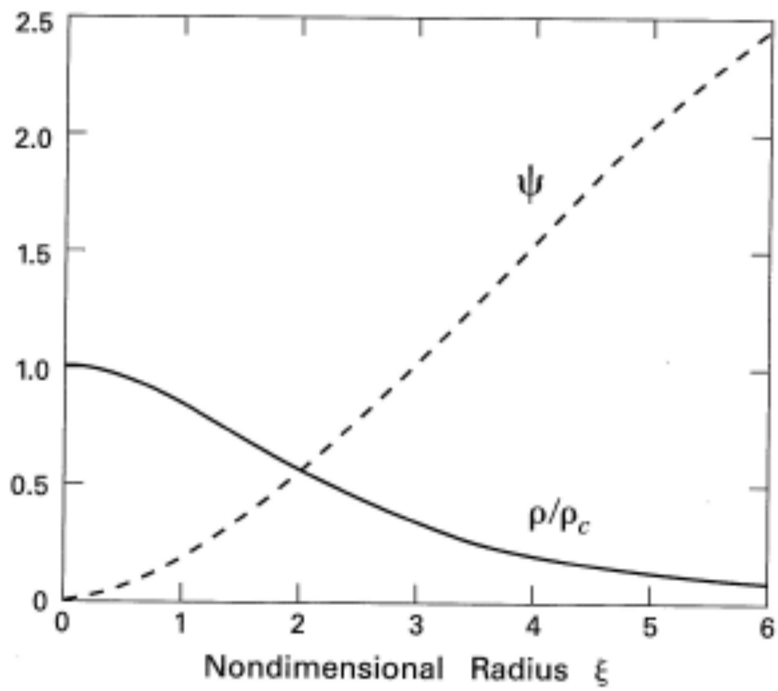

Figure 9.1 – Numerical solutions for the isothermal Lane-Emden equation for the dimensionless gravitational potential $\psi(\xi)$ (broken curves) and the mass density (solid curve) of an isothermal sphere.

The fact that the density decreases away from the centre is to be expected from the opposite behavior of the potential. This is required for hydrostatic equilibrium in equation (9.1) and is not limited to system exhibiting spherical symmetry. Since the pressure scales proportionally to the density, we have that $-\nabla p > 0$, which is required to counterbalance the gravitational potential (i.e., $\nabla\Phi_{\mathrm{g}} < 0$).

It is also worth noting that the density of an isothermal sphere does not asymptotically go to zero, a behavior that may seem unphysical. This is, however, not a problem when approximating real molecular clouds with such models. This is because we can set the pressure p_0 (or the density ρ_0 from equation (9.2)) at the boundary of the cloud equal to that of the surrounding interstellar medium. Moreover, observations (or any theoretical model under consideration) may set the density ρ_{c} at the centre of the cloud. It follows that Figure 9.1 will allow us to determine the dimensionless radius ξ_0 (or the radius r_0 from the last of equations (9.10)) to be used for the cloud. The peculiar asymptotic behavior of the isothermal sphere is therefore not a problem in that respect. Furthermore, once r_0 is determined it is possible to estimate the mass of the sphere with

$$\begin{aligned} M &= 4\pi\int_0^{r_0} \rho r^2 dr \\ &= 4\pi\rho_{\mathrm{c}}\left(\frac{a_T^2}{4\pi G\rho_{\mathrm{c}}}\right)^{-3/2} \int_0^{\xi_0} e^{-\psi}\xi^2 d\xi. \end{aligned} \tag{9.12}$$

This equation is further transformed with the realization from equation (9.11) that

$$\begin{aligned}\int_0^{\xi_0} e^{-\psi}\xi^2 d\xi &= \int_0^{\xi_0} d\left(\xi^2 \frac{d\psi}{d\xi}\right)\\ &= \xi^2 \frac{d\psi}{d\xi}\bigg|_{\xi_0},\end{aligned} \tag{9.13}$$

which yields after insertion in equation (9.12)

$$\begin{aligned}m &\equiv \frac{p_0^{1/2}G^{3/2}M}{a_T^4}\\ &= \left(4\pi\frac{\rho_c}{\rho_0}\right)^{-1/2}\left(\xi^2\frac{d\psi}{d\xi}\right)_{\xi_0},\end{aligned} \tag{9.14}$$

with m is the dimensionless cloud mass. A numerical solution of equation (9.14), using the results of Figure 9.1, is shown in Figure 9.2.

Incidentally, the aforementioned asymptotic value for the density is given by

$$\lim_{\xi\to\infty}\frac{\rho}{\rho_c} = \frac{2}{\xi^2}, \tag{9.15}$$

with an associated potential (from equation (9.8))

$$\lim_{\xi\to\infty}\psi = \ln\left(\frac{\xi^2}{2}\right). \tag{9.16}$$

It is interesting to note that this potential, when taken for any values of ξ (i.e., not just asymptotically), also satisfies the isothermal Lane-Emden equation (i.e., equation (9.11)), but not the boundary conditions $\psi(0) = d\psi/d\xi\big|_{\xi=0} = 0$. It cannot, therefore, be an acceptable physical solution. The associated density profile is that of the so-called **singular isothermal sphere**.

$$\rho(r) = \frac{a_T^2}{2\pi G r^2}. \tag{9.17}$$

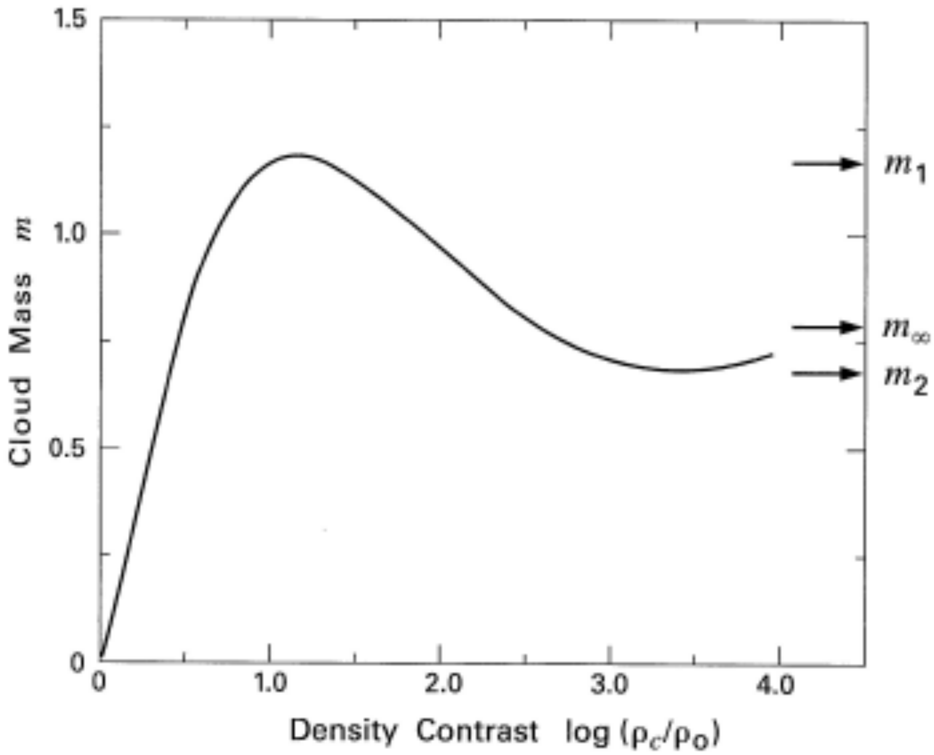

Figure 9.2 – Numerical solution for the dimensionless mass of pressure-bounded isothermal spheres as a function of the density contrast from the centre to the outer edge.

9.1.1 Gravitational Stability

Inspection of Figure 9.2 can help in qualitatively determine the conditions that lead to gravitational stability or instability. Let us consider, for example, the region of the curve located to the left of the first maximum (and only one on this graph), where $m \equiv m_1$. To do so we investigate how the cloud will react to a perturbation applied to its boundary. More precisely, any increase in the pressure p_0 at the boundary of the cloud will be met by a corresponding increase in the density ρ_0. Since on the other hand the mass M of the cloud must remain unchanged, and we are dealing with an isothermal sphere (i.e., a_T is constant), then the first of equations (9.14) tells us that m must increase. This implies that the increase in p_0 causes us to move to the right of the curve of Figure 9.2, and that, therefore, the density ρ_c at the centre of the cloud increases in a bigger proportion than ρ_0. But we know from the behavior of the density, and therefore the pressure, in Figure 9.1 that the cloud will react by increasing the pressure gradient in its interior. We can see this by considering the total change in pressure from the centre to the boundary becomes

$$\begin{aligned}\delta\left(p_c - p_0\right) &= a_T^2 \delta\left(\rho_c - \rho_0\right) \\ &= a_T^2 \delta\rho_0 \left(\frac{\delta\rho_c}{\delta\rho_0} - 1\right) > 0, \qquad \text{when } m < m_1\end{aligned} \tag{9.18}$$

since we are moving on the left portion of the curve of Figure 9.2 where the slope is positive. Equation (9.18) tells us that the cloud will not collapse when subjected to a pressure increase (at the perturbation level, that is) at its boundary; it is thus gravitationally stable.

Evidently, equation (9.18) also makes it clear that the reverse will be true when we consider a cloud that is initially located somewhere on the right of $m = m_1$ on the curve of Figure 9.2, where the slope is negative. That is,

$$\delta\left(p_c - p_0\right) = a_T^2 \delta\rho_0 \left(\frac{\delta\rho_c}{\delta\rho_0} - 1\right) < 0, \qquad \text{when } m > m_1. \tag{9.19}$$

This cloud is therefore unstable to the perturbation and gravitationally unstable. We thus find from the first of equations (9.14) the mass that differentiates stable from unstable isothermal sphere at a given temperature and boundary pressure. This is the so-called **Bonner-Ebert mass**

$$M_{\mathrm{BE}} = \frac{m_1 a_T^4}{p_0^{1/2} G^{3/2}}, \tag{9.20}$$

with $m_1 = 1.18$.

9.1.2 The Jeans Length and Mass

We now proceed with a more quantitative analysis of the stability problem by considering an isothermal gas of initial uniform density ρ_0, which is subjected to a perturbation such that

$$\rho\left(x,t\right) = \rho_0 + \delta\rho e^{i(kx - \omega t)}. \tag{9.21}$$

This type of one-dimensional sinusoidal perturbation can always be generalized to any arbitrary function by the use of a Fourier series or transform. But it is perfectly suited for the type of idealized gas considered here. We further assume that a similar perturbation also exists for the pressure. The gravitational potential $\delta\Phi_g$ and the velocity δu (in the x-direction) only result from the perturbation. The system of equations to be analyzed is composed of

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) &= 0 \\
p &= a_T^2 \rho \\
\frac{\partial^2 \Phi_g}{\partial x^2} &= 4\pi G \rho \\
\rho \frac{du}{dt} &= -\frac{\partial p}{\partial x} - \rho \frac{\partial \Phi_g}{\partial x}.
\end{aligned}
\tag{9.22}
$$

Taking into account the perturbations for the different parameters, while keeping calculations to first order, these equations become

$$
\begin{aligned}
-i\omega \delta\rho + ik\rho_0 \delta u &= 0 \\
\delta p &= a_T^2 \delta\rho \\
-k^2 \delta\Phi_g &= 4\pi G \delta\rho \\
-i\omega\rho_0 \delta u &= -ik\delta p - ik\rho_0 \delta\Phi_g.
\end{aligned}
\tag{9.23}
$$

The velocity can be eliminated from the last equation by inserting the first into it

$$
\begin{aligned}
\omega^2 \delta\rho &= k^2 \left(\delta p + \rho_0 \delta\Phi_g\right) \\
&= k^2 \left(a_T^2 \delta\rho + \rho_0 \delta\Phi_g\right).
\end{aligned}
\tag{9.24}
$$

We now eliminate the gravitational potential with the third of equations (9.23)

$$
\left(\omega^2 - a_T^2 k^2 + 4\pi G \rho_0\right)\delta\rho = 0, \tag{9.25}
$$

an equation that can only be true in general if

$$
\omega^2 - a_T^2 k^2 + 4\pi G \rho_0 = 0. \tag{9.26}
$$

This is the **dispersion equation** for this problem; the corresponding curve is shown in Figure 9.3. The roots for this equation are

$$
\omega = \pm k \sqrt{a_T^2 - \frac{4\pi G \rho_0}{k^2}}. \tag{9.27}
$$

We therefore find that wave numbers that satisfy

$$
k^2 < \frac{4\pi G \rho_0}{a_T^2} \equiv \frac{\omega_0^2}{a_T^2} \equiv k_0^2 \tag{9.28}
$$

imply potentially unstable perturbations, as the exponential in equation (9.21) has a real exponent. The critical wavelength where this behavior arises is

$$\begin{aligned}\lambda_{\mathrm{J}} &\equiv \left(\frac{\pi a_T^2}{G\rho_0}\right)^{1/2} \\ &= 0.19\left(\frac{T}{10\ \mathrm{K}}\right)^{1/2}\left(\frac{n_{\mathrm{H}_2}}{10^4\ \mathrm{cm}^{-3}}\right)^{-1/2}\ \mathrm{pc}.\end{aligned} \tag{9.29}$$

This is the definition for the so-called **Jeans length**. It is to be expected that the Jeans length must be related to the size of the cloud, as the wavelength of the perturbation should approximately fit its boundary. We calculate the corresponding mass enclosed

$$\begin{aligned}M &= 4\pi\rho_0\int_0^{\lambda_{\mathrm{J}}/2} r^2 dr \\ &= \frac{\pi\rho_0}{6}\left(\frac{\pi a_T^2}{G\rho_0}\right)^{3/2} \\ &= \frac{\pi}{6}\frac{M_{\mathrm{BE}}}{m_1}.\end{aligned} \tag{9.30}$$

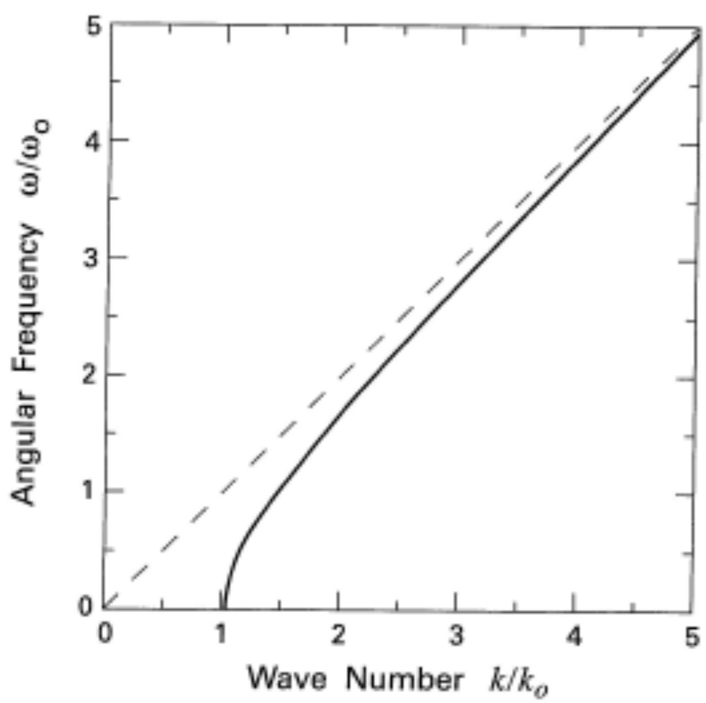

Figure 9.3 – The dispersion curve for a self-gravitating and initially uniform isothermal sphere.

Because of this the Bonnor-Ebert mass is more commonly known as the **Jeans mass**, when expressed as a function of the mass density instead of the pressure

$$\begin{aligned} M_{\mathrm{J}} &\equiv \frac{m_1 a_T^3}{G^{3/2}\rho_0^{1/2}} \\ &= 1.0\left(\frac{T}{10\ \mathrm{K}}\right)^{3/2}\left(\frac{n_{\mathrm{H_2}}}{10^4\ \mathrm{cm^{-3}}}\right)^{-1/2} M_{\odot}. \end{aligned} \tag{9.31}$$

We therefore see from equation (9.31) that dense cores and Bok globules are on the verge of instability, which accounts well for the fact that some of them harbor protostars. On the other hand, larger clumps with, say, $T \approx 10\ \mathrm{K}$ and $n_{\mathrm{H_2}} \approx 10^3\ \mathrm{cm^{-3}}$ have $M_{\mathrm{J}} \approx 3M_{\odot}$, which is a mass that is a few orders of magnitude lower than what is measured. It therefore follows that there must exists at least another mechanism for support against gravitation, as our analysis reveals that thermal pressure could not be enough for this.

9.2 Magnetostatic Configurations

We now include the presence of magnetic field lines threading the gas of our isothermal, non-rotating clouds. In this case, the equation of balance becomes

$$-a_T^2\nabla\rho - \rho\nabla\Phi_{\mathrm{g}} + \frac{1}{c}\mathbf{j}\times\mathbf{B} = 0 \tag{9.32}$$

with the addition of the Lorentz force. To simplify the problem at hand we will also assume that our model cloud is threaded by a poloidal magnetic field whose main component is directed along the z-axis, and possesses cylindrical and reflection (about the plane $z = 0$) symmetries. We will use the (ϖ,ϕ,z) cylindrical coordinate system.

Since the magnetic field can be expressed with the vector potential $\mathbf{A}$ as

$$\mathbf{B} = \nabla\times\mathbf{A}, \tag{9.33}$$

then magnetic flux $\mathbf{\Phi}_B$ is written as follows

$$\begin{aligned} \Phi_B &= \int_S \mathbf{B}\cdot\mathbf{n}\,da \\ &= \int_S (\nabla\times\mathbf{A})\cdot\mathbf{n}\,da \\ &= \oint_C \mathbf{A}\cdot d\mathbf{l} \\ &= 2\pi\varpi A_\phi, \end{aligned} \tag{9.34}$$

with S and C are the open surface of integration and its contour, $\mathbf{n}$ is the unit vector normal to the surface, and where Stokes' theorem and the assumed cylindrical symmetry of the field were used. We see from this calculation that the potential only has an azimuthal component. Using the following identity (which we prove)

$$\begin{aligned}
\mathbf{b}_i &= \left[\nabla\times(\chi\mathbf{c})\right]_i \\
&= \varepsilon_{ijk}\,\partial_j\left(\chi c_k\right) \\
&= \varepsilon_{ijk}\left(c_k\,\partial_j\chi + \chi\,\partial_j c_k\right) \\
&= \left[(\nabla\chi)\times\mathbf{c} + \chi(\nabla\times\mathbf{c})\right]_i,
\end{aligned} \tag{9.35}$$

we find that

$$\begin{aligned}
\mathbf{B} &= \nabla\times\left(A_\phi\mathbf{e}_\phi\right) \\
&= \left(\nabla A_\phi\right)\times\mathbf{e}_\phi + A_\phi\left(\nabla\times\mathbf{e}_\phi\right) \\
&= \left(\nabla A_\phi\right)\times\mathbf{e}_\phi + A_\phi\frac{\mathbf{e}_z}{\varpi} \\
&= \left(\nabla A_\phi\right)\times\mathbf{e}_\phi + \frac{A_\phi}{\varpi}\left(\mathbf{e}_\varpi\times\mathbf{e}_\phi\right) \\
&= \left(\nabla A_\phi + \frac{A_\phi}{\varpi}\mathbf{e}_\varpi\right)\times\mathbf{e}_\phi.
\end{aligned} \tag{9.36}$$

But since A_ϕ can only be a function of ϖ and z, we then have

$$\begin{aligned}
\mathbf{B} &= \frac{\nabla\left(\varpi A_\phi\right)}{\varpi}\times\mathbf{e}_\phi \\
&= -\frac{\mathbf{e}_\phi}{2\pi\varpi}\times\nabla\Phi_B,
\end{aligned} \tag{9.37}$$

which implies that $\mathbf{B}\cdot\nabla\Phi_B = 0$ and that the magnetic flux can only vary in directions perpendicular to the magnetic field.

We also need to consider the vectorial nature of the current density $\mathbf{j}$. Because of the symmetry of the problem the magnetic field must be of the form

$$\mathbf{B} = B_\varpi\mathbf{e}_\varpi + B_z\mathbf{e}_z, \tag{9.38}$$

i.e., it does not have an azimuthal component. It follows that the integral form of Ampère's law can be written as

$$\oint_{C'} \mathbf{B} \cdot d\mathbf{l} = \int_{S'} (\nabla \times \mathbf{B}) \cdot \mathbf{e}_\phi \, da$$
$$= \frac{4\pi}{c} \int_{S'} \mathbf{j} \cdot \mathbf{e}_\phi \, da \tag{9.39}$$
$$= \frac{4\pi}{c} j_\phi \, \Delta S',$$

Since the surface of integration S' (of area ΔS) and its contour C' can be chosen to lie in the ϖz-plane without loss of generality. It follows from equation (9.39) that, like the vector potential, the current density only has an azimuthal component that can only be a function of ϖ and z.

Let us further consider the quantity

$$\begin{aligned} q &\equiv a_T^2 \rho e^{\Phi_\mathrm{g}/a_T^2} \\ &= p e^{\Phi_\mathrm{g}/a_T^2}, \end{aligned} \tag{9.40}$$

which is similar to the pressure for the magnetic free isothermal sphere case of the previous section, and its gradient

$$\nabla q = e^{\Phi_\mathrm{g}/a_T^2} \nabla p + \frac{p}{a_T^2} e^{\Phi_\mathrm{g}/a_T^2} \nabla \Phi_\mathrm{g}. \tag{9.41}$$

We insert this relation and equation (9.37) into equation (9.32), while keeping equation (9.39) in consideration

$$\begin{aligned} -a_T^2 \nabla p - \rho \nabla \Phi_\mathrm{g} + \frac{1}{c} \mathbf{j} \times \mathbf{B} &= -e^{-\Phi_\mathrm{g}/a_T^2} \nabla q - \frac{j_\phi}{c} \left[\mathbf{e}_\phi \times \left(\frac{\mathbf{e}_\phi}{2\pi\varpi} \times \nabla \Phi_B \right) \right] \\ &= -e^{-\Phi_\mathrm{g}/a_T^2} \nabla q - \frac{j_\phi}{c} \left[\frac{\mathbf{e}_\phi}{2\pi\varpi} \left(\mathbf{e}_\phi \cdot \nabla \Phi_B \right) - \left(\frac{\mathbf{e}_\phi}{2\pi\varpi} \cdot \mathbf{e}_\phi \right) \nabla \Phi_B \right] \\ &= -e^{-\Phi_\mathrm{g}/a_T^2} \nabla q + \frac{j_\phi}{2\pi\varpi c} \nabla \Phi_B \\ &= 0, \end{aligned} \tag{9.42}$$

or

$$e^{-\Phi_\mathrm{g}/a_T^2} \nabla q = \frac{j_\phi}{2\pi\varpi c} \nabla \Phi_B. \tag{9.43}$$

Because of the fact that $\mathbf{B} \cdot \nabla \Phi_B = 0$ we find that

$$\mathbf{B} \cdot \nabla q = 0, \tag{9.44}$$

and that

$$q = q\left(\Phi_B\right). \tag{9.45}$$

It is therefore reasonable to transform equation (9.43) to

$$\frac{j_\phi}{2\pi\varpi c} = \frac{dq}{d\Phi_B} e^{-\Phi_g/a_T^2}. \tag{9.46}$$

We now seek to replace the azimuthal current density component in equation (9.46) with Ampère's law using equation (9.37)

$$\begin{aligned}
j_\phi &= \frac{c}{4\pi}\left(\nabla\times\mathbf{B}\right)_\phi \\
&= \frac{c}{4\pi}\left[\nabla\times\left(\nabla\Phi_B\times\frac{\mathbf{e}_\phi}{2\pi\varpi}\right)\right]_\phi \\
&= \frac{c}{4\pi}\left[\left(\nabla\cdot\frac{\mathbf{e}_\phi}{2\pi\varpi}\right)\nabla\Phi_B - \frac{\mathbf{e}_\phi}{2\pi\varpi}\nabla^2\Phi_B\right]_\phi \\
&= -\frac{c}{8\pi^2\varpi}\left[\frac{1}{\varpi}\frac{\partial}{\partial\varpi}\left(\varpi\frac{\partial\Phi_B}{\partial\varpi}\right) + \frac{\partial^2\Phi_B}{\partial z^2}\right].
\end{aligned} \tag{9.47}$$

Insertion in equation (9.46) we find

$$\frac{1}{\varpi}\frac{\partial}{\partial\varpi}\left(\varpi\frac{\partial\Phi_B}{\partial\varpi}\right) + \frac{\partial^2\Phi_B}{\partial z^2} = -16\pi^3\varpi\frac{dq}{d\Phi_B}e^{-\Phi_g/a_T^2}. \tag{9.48}$$

We can rewrite Poisson's equation in exactly the same manner

$$\frac{1}{\varpi}\frac{\partial}{\partial\varpi}\left(\varpi\frac{\partial\Phi_g}{\partial\varpi}\right) + \frac{\partial^2\Phi_g}{\partial z^2} = \frac{4\pi G}{a_T^2}qe^{-\Phi_g/a_T^2}. \tag{9.49}$$

Equations (9.48) and (9.49) are two coupled relations that we need to simultaneously solve to obtain Φ_B and Φ_g. But to do so the quantity $q\left(\Phi_B\right)$ and, therefore, the magnetic flux distribution must be specified. This is done by first calculating the amount of mass ΔM contain between the radii where the flux goes from Φ_B and $\Phi_B + \Delta\Phi_B$ with (see Figure 9.4)

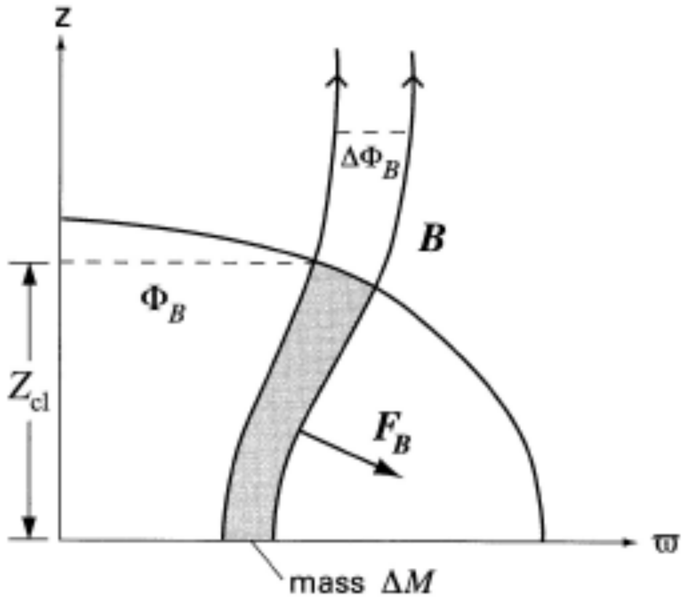

Figure 9.4 – One quadrant of a magnetically supported cloud.

$$\begin{aligned}\Delta M &= 2\int_0^{Z_c(\Phi_B)} dz \int_{\varpi(z,\Phi_B)}^{\varpi(z,\Phi_B+\Delta\Phi_B)} 2\pi\rho\varpi d\varpi \\ &= 4\pi\int_0^{Z_c(\Phi_B)} dz\,\rho\Delta\Phi_B\varpi\frac{\partial\varpi}{\partial\Phi_B},\end{aligned} \tag{9.50}$$

where $Z_c(\Phi_B)$ is the cloud boundary and a change of variable from ϖ to Φ_B was made. Eliminating the mass density with equation (9.40) we finally find that

$$q = \frac{a_T^2}{4\pi}\frac{dM}{d\Phi_B}\left[\int_0^{Z_c(\Phi_B)} dz\,\varpi\frac{\partial\varpi}{\partial\Phi_B}e^{-\Phi_g/a_T^2}\right]^{-1}. \tag{9.51}$$

This equation and its derivative are to be inserted in equations (9.48) and (9.49).

However, in deriving equation (9.51) we have introduce yet another quantity, $dM/d\Phi_B$, which must be specified. This is usually done through some idealized and simple representation. For example, if the cloud has contracted from a sphere of uniform density ρ_i and radius R_0 threaded by a background magnetic field B_0, then proceeding as we did for equation (9.50) when integrating within the confine of the cloud we find that

$$\begin{aligned}dM &= 2\int_0^{\sqrt{R_0^2-\varpi^2}} dz\,2\pi\varpi\rho\,d\varpi \\ &= 2\int_0^{\sqrt{R_0^2-\varpi^2}} dz\frac{\rho}{B_0}d\Phi_B,\end{aligned} \tag{9.52}$$

and

$$\frac{dM}{d\Phi_B} = \frac{2\rho R_0}{B_0}\left(1 - \frac{\varpi^2}{R_0^2}\right)^{1/2}$$
$$= \frac{2\rho R_0}{B_0}\left(1 - \frac{\Phi_B}{\Phi_c}\right)^{1/2}, \tag{9.53}$$

where $\Phi_c \equiv \pi R_0^2 B_0$ is the total flux threading the cloud. Evidently, the amount of mass accrued must cease when the boundary of the cloud is reached. We then write

$$\frac{dM}{d\Phi_B} = \begin{cases} \frac{2\rho R_0}{B_0}\left(1 - \frac{\Phi_B}{\Phi_c}\right)^{1/2} & \Phi_B \leq \Phi_c \\ 0, & \Phi_B > \Phi_c. \end{cases} \tag{9.54}$$

If we assume that this relation holds during the evolution of the cloud (i.e., flux freezing), then equation (9.54) is to be inserted in equation (9.51) and the problem can be solved numerically. Results are shown in Figure 9.5 where the three dimensionless quantities used are defined as

$$\begin{aligned} &\frac{\rho_c}{\rho_0}, && \text{density contrast} \\ &\alpha \equiv \frac{B_0^2}{8\pi p_0}, && \text{magnetic to thermal pressure ratio (background medium)} \\ &\xi_0 \equiv \left(\frac{4\pi G\rho_0}{a_T^2}\right)^{1/2} R_0, && \text{dimensionless initial radius.} \end{aligned} \tag{9.55}$$

As is clear from the figure, the resulting clouds, either stable or not, are significantly flattened along the axis of symmetry (the field lines).

9.2.1 **The Critical Mass**

Once models such as the one presented in Figure 9.5 are in hand, we can use equation (9.52) to calculate the mass, and consequently the dimensionless mass defined with the first of equations (9.14).

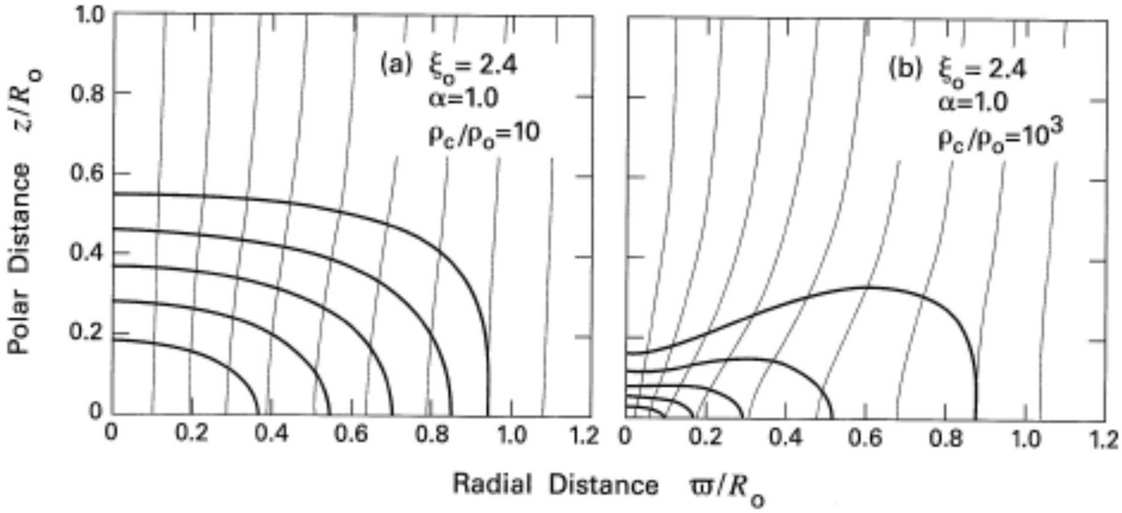

Figure 9.5 – a) Density contours (thick curves) for a cloud that is magnetically supported and gravitationally stable. b) Density contours for an unstable cloud. The thin curves are for the magnetic field lines in both cases.

Curves of the dimensionless mass as a function of the density contrast are shown in Figure 9.6 for different values for the parameters listed in equations (9.55). As was the case for non-magnetized clouds, there exists a set of critical mass beyond which the cloud is gravitationally unstable. This mass can be numerically fitted with

$$m_{\text{crit}} \approx 1.2 + 0.15\alpha^{1/2}\xi_0^2, \tag{9.56}$$

or equivalently

$$M_{\text{crit}} \approx M_{\text{BE}} + M_\Phi, \tag{9.57}$$

with

$$\begin{aligned} M_\Phi &= 0.15\frac{\alpha^{1/2}\xi_0^2 a_T^4}{p_0^{1/2}G^{3/2}} \\ &= 0.15\frac{2}{\sqrt{2\pi}}\frac{\pi R_0^2 B_0}{G^{1/2}} \\ &= 0.12\frac{\Phi_{\text{c}}}{G^{1/2}}. \end{aligned} \tag{9.58}$$

A cloud with $M < M_\Phi$ can never undergo gravitational collapse (if there is no ambipolar diffusion), since it is only a function of the magnetic flux threading the cloud (unlike the Jeans or Bonnor-Ebert mass, which is a function of the pressure at the boundary of the cloud). Equation (9.58) can be advantageously transformed to

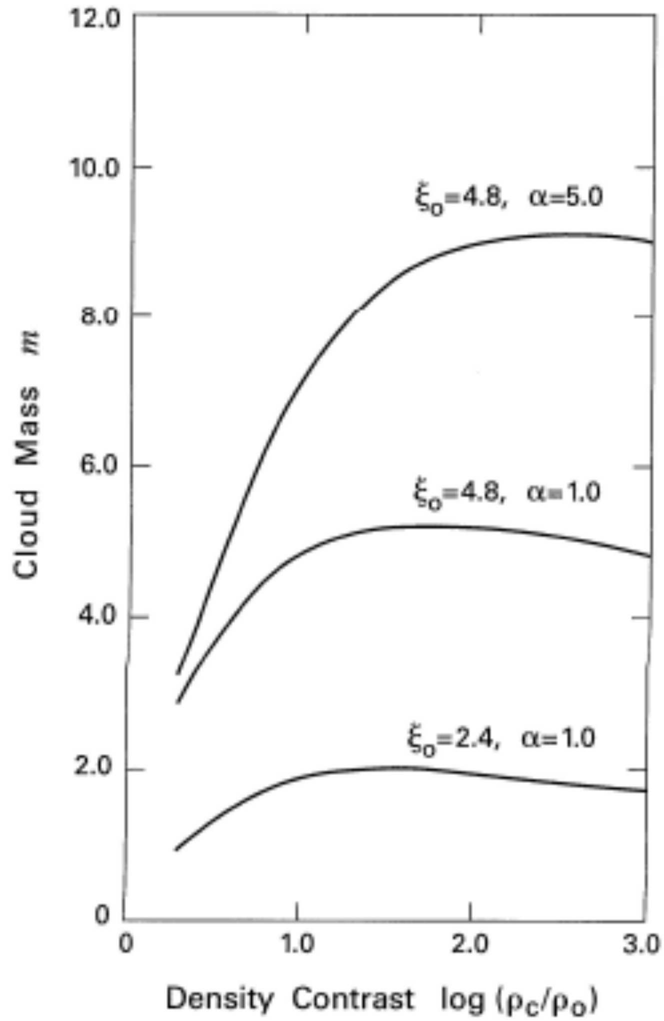

Figure 9.6 – Dimensionless mass plotted as a function of the density contrast for magnetized clouds.

$$M_{\Phi} = 70\left(\frac{B}{10\,\mu\text{G}}\right)\left(\frac{R}{10\text{ pc}}\right)^2 M_{\odot}, \tag{9.59}$$

where B is the mean magnetic field threading the cloud. This relation seems to account for the fact that clouds or clumps much more massive than the Jeans mass are not undergoing gravitational collapse. Magnetic support can readily bring a satisfying explanation to this apparent problem. It also does not change the previous scenario discussed for dark clouds and Bok globules since these entities are primarily thermally supported. In these cases we find that $M \approx M_{\text{BE}} \approx M_{\Phi}$. All is not perfect, however, as the highly flattened profiles predicted by this model (and shown in Figure 9.5) are not observed. We must, therefore, find a new mechanism that can provide further support align the fields line.

9.3 Support from Magnetohydrodynamics (MHD) Waves

We know from Problem 5 of the First Assignment that the equation of motion for the neutral component of a weakly ionized medium (e.g., the gas within a molecular cloud) is given by (neglecting gravitation, as we will consider variation in the different parameters that are much smaller than the Jeans length)

$$\rho\left[\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}\right]=-\nabla p+\frac{1}{4\pi}(\nabla\times\mathbf{B})\times\mathbf{B}. \tag{9.60}$$

To this equation we must add a few more equations. Two of these are the continuity and the state equations

$$\begin{aligned}\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\mathbf{u})=0\\ p=a_T^2\rho,\end{aligned} \tag{9.61}$$

while the last one is derived as follows. We start with Ohm's law for such a medium (neglecting diffusive processes)

$$\mathbf{E}=-\frac{\mathbf{u}}{c}\times\mathbf{B}, \tag{9.62}$$

which we insert into Faraday's law

$$\begin{aligned}\frac{\partial \mathbf{B}}{\partial t}&=-c\nabla\times\mathbf{E}\\ &=\nabla\times(\mathbf{u}\times\mathbf{B}).\end{aligned} \tag{9.63}$$

This last relation is the **induction equation**, which can be used to prove the **flux-freezing** phenomenon, for example. Equations (9.60), (9.61), and (9.63) form a complete set of relations that should be solved simultaneously.

9.3.1 Hydromagnetic Waves

We now consider perturbations on an initially uniform and homogeneous plasma fluid, and we write the density, pressure, velocity, and magnetic field as $\rho_0+\delta\rho(\mathbf{x},t)$, $p_0+\delta p(\mathbf{x},t)$, $\delta\mathbf{u}(\mathbf{x},t)$, and $\mathbf{B}_0+\delta\mathbf{B}(\mathbf{x},t)$, respectively. Also, we will assume that we are dealing with the more general adiabatic fluid (i.e., the perturbations in density and pressure evolve on times short compared to the typical time required for heat conduction, and we can approximate $dQ=0$, where Q is the heat) with

$$p=K\rho^{\gamma}, \tag{9.64}$$

where K is a constant and γ is the ratio of the specific heats (for the isothermal gas $\gamma = 1$ and $K = a_T^2$). We could simply modify this relation as follows

$$p = K\rho_0^\gamma \left(1 + \frac{\delta\rho}{\rho_0}\right)^\gamma \simeq K\rho_0^\gamma \left(1 + \gamma\frac{\delta\rho}{\rho_0}\right), \tag{9.65}$$

or

$$\delta p = K\gamma\rho_0^{\gamma-1}\delta\rho = c_s^2\delta\rho, \tag{9.66}$$

with

$$c_s^2 = \frac{\gamma p_0}{\rho_0}, \tag{9.67}$$

and then insert it in the equation of motion (i.e., equation (9.60); again, for the isothermal gas $c_s = a_T$). But it is to our benefit to consider the behavior of such a fluid without the presence of a magnetic induction to get a better understanding of the corresponding terms. So, keeping things to the first order in the perturbations, we transform the continuity equation to

$$\frac{\partial(\delta\rho)}{\partial t} + \rho_0\nabla\cdot\delta\mathbf{u} = 0, \tag{9.68}$$

while the equation of motion (with $\mathbf{B} = 0$) becomes

$$\rho_0\frac{\partial(\delta\mathbf{u})}{\partial \mathrm{t}} + \nabla(\delta p) = 0. \tag{9.69}$$

Inserting equation (9.66) in equation (9.69) we find

$$\rho_0\frac{\partial(\delta\mathbf{u})}{\partial t} + c_s^2\nabla(\delta\rho) = 0, \tag{9.70}$$

which we put into equation (9.68) (after having taken its time derivative) to obtain

$$\frac{\partial^2(\delta\rho)}{\partial t^2} - c_s^2\nabla^2(\delta\rho) = 0. \tag{9.71}$$

Equation (9.71) is a wave equation that describes the propagation of (longitudinal) acoustic perturbations in the fluid. Obviously, the parameter c_s is nothing else but the **sound speed** for the medium, which is generally expressed by

$$c_s^2 = \frac{dp}{d\rho}. \tag{9.72}$$

Proceeding further, we now turn to the case of a weakly ionized plasma. We are in a position to write down the set of linearized magnetohydrodynamics equations (from equation (9.60), the first of equations (9.61), and (9.63)) that characterize the behavior of the plasma under consideration

$$\begin{aligned} \frac{\partial(\delta\rho)}{\partial t} + \rho_0 \nabla \cdot \delta\mathbf{u} = 0 \\ \rho_0 \frac{\partial(\delta\mathbf{u})}{\partial t} + c_s^2 \nabla(\delta\rho) + \frac{1}{4\pi}\mathbf{B}_0 \times \left(\nabla \times (\delta\mathbf{B})\right) = 0 \\ \frac{\partial(\delta\mathbf{B})}{\partial t} - \nabla \times (\delta\mathbf{u} \times \mathbf{B}_0) = 0. \end{aligned} \tag{9.73}$$

On taking the time derivative of the second of equations (9.73), and inserting in it the other two equations of the same set, we find

$$\frac{\partial^2(\delta\mathbf{u})}{\partial t^2} - c_s^2 \nabla(\nabla \cdot \delta\mathbf{u}) + \mathbf{v}_A \times \left\{\nabla \times \left[\nabla \times (\delta\mathbf{u} \times \mathbf{v}_A)\right]\right\} = 0, \tag{9.74}$$

where we have introduced the so-called **Alfvén velocity**

$$\mathbf{v}_A = \frac{\mathbf{B}_0}{\sqrt{4\pi\rho_0}}. \tag{9.75}$$

Although it may not necessarily be obvious at this point, equation (9.74) is also a wave equation. We can verify this by considering an arbitrary Fourier component $\delta\mathbf{u}\, e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}$ for the velocity field. Inserting this function in equation (9.74) yields

$$\begin{aligned} \omega^2 \delta\mathbf{u} &= c_s^2 \mathbf{k}(\mathbf{k} \cdot \delta\mathbf{u}) - \mathbf{v}_A \times \left\{\mathbf{k} \times \left[\mathbf{k} \times (\delta\mathbf{u} \times \mathbf{v}_A)\right]\right\} \\ &= c_s^2 \mathbf{k}(\mathbf{k} \cdot \delta\mathbf{u}) - \mathbf{v}_A \times \left\{\mathbf{k} \times \left[(\mathbf{k} \cdot \mathbf{v}_A)\delta\mathbf{u} - (\mathbf{k} \cdot \delta\mathbf{u})\mathbf{v}_A\right]\right\} \\ &= c_s^2 \mathbf{k}(\mathbf{k} \cdot \delta\mathbf{u}) - \mathbf{k}\left[(\mathbf{k} \cdot \mathbf{v}_A)(\delta\mathbf{u} \cdot \mathbf{v}_A) - (\mathbf{k} \cdot \delta\mathbf{u})v_A^2\right] \\ &\quad + (\mathbf{k} \cdot \mathbf{v}_A)\left[(\mathbf{k} \cdot \mathbf{v}_A)\delta\mathbf{u} - (\mathbf{k} \cdot \delta\mathbf{u})\mathbf{v}_A\right], \end{aligned} \tag{9.76}$$

and finally

$$\omega^2 \delta\mathbf{u} = \left(c_s^2 + v_A^2\right)\mathbf{k}(\mathbf{k} \cdot \delta\mathbf{u}) + (\mathbf{k} \cdot \mathbf{v}_A)\left[(\mathbf{k} \cdot \mathbf{v}_A)\delta\mathbf{u} - (\mathbf{k} \cdot \delta\mathbf{u})\mathbf{v}_A - (\delta\mathbf{u} \cdot \mathbf{v}_A)\mathbf{k}\right]. \tag{9.77}$$

This last equation is the basic dispersion relation for hydromagnetic waves relating $\mathbf{k}$ and ω when c_{s} and v_{A} are known. Although the solution of this equation is not straightforward, it allows for three different modes of propagation (see the Appendix at the end of this chapter), of which one is easily determined. This simple solution is that of transverse waves perpendicular to the plane defined by the propagation vector $\mathbf{k}$ and the Alfvén velocity $\mathbf{v}_{\mathrm{A}}$. That is, setting $\mathbf{k}\cdot\delta\mathbf{u} = \delta\mathbf{u}\cdot\mathbf{v}_{\mathrm{A}} = 0$ we find

$$\omega^2 = \left(\mathbf{k}\cdot\mathbf{v}_{\mathrm{A}}\right)^2, \tag{9.78}$$

so that

$$\omega = \pm v_{\mathrm{A}} k \cos(\theta), \tag{9.79}$$

where θ is the angle between the magnetic field and the wave vector. The waves defined by equation (9.79) are called **Alfvén waves**, and suggest the picture of a magnetized plasma akin to that of a string on which transverse waves propagate. Inspection of first and third of equations (9.73) yield

$$\begin{aligned}
\delta\rho &= 0 \\
\delta\mathbf{B} &= -\frac{\mathbf{k}}{\omega}\times\left(\delta\mathbf{u}\times\mathbf{B}_0\right) \\
&= \frac{1}{\omega}\left[\left(\mathbf{k}\cdot\delta\mathbf{u}\right)\mathbf{B}_0 - \left(\mathbf{k}\cdot\mathbf{B}_0\right)\delta\mathbf{u}\right] \\
&= -\frac{k}{\omega}B_0\cos(\theta)\delta\mathbf{u}
\end{aligned} \tag{9.80}$$

The first of equations (9.80) implies that there will be no compression of the gas, as is the case for hydrodynamic waves. This is an interesting property because it implies that these waves will not produce shocks in the gas, and therefore will not dissipate quickly. In other words, Alfvén waves are good candidates for supporting the cloud on a longer time scale. The last of equations (9.80) reveals that perturbations on the magnetic field are perpendicular to the background field $\mathbf{B}_0$. Following this line of thought, we see that any transverse perturbation on the magnetic field lines would be opposed by the magnetic tension. It is also noted from equation (9.79) that the group velocity of the wave is $v_{\mathrm{A}}\cos(\theta)$.

The other two modes of propagation are the so-called **fast** and **slow modes**, which can be shown to exhibit compression in the gas, in general. They are therefore subject to dissipation. The Alfvén are thus the best candidate for providing support against gravitation. Examples of the different mode of wave propagation are shown in Figure 9.7.

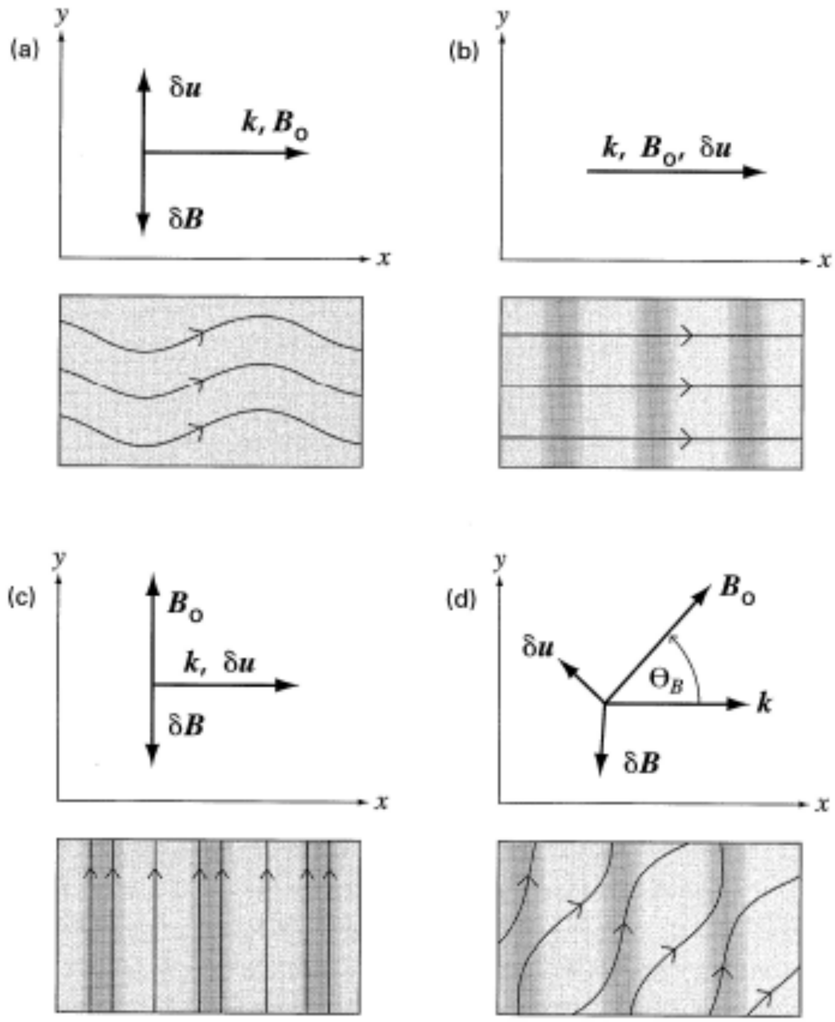

Figure 9.7 – Examples of MHD waves for different orientations for background magnetic field, wave propagation, velocity and magnetic field perturbations. Case (a) and (d) correspond to Alfvén and fast/slow mode waves, respectively.

Finally, we already know that the magnetic pressure is given by

$$p_{\text{mag}} = \frac{B^2}{8\pi}, \tag{9.81}$$

and the associate force density by

$$\mathbf{f}_{\text{mag}} = -\nabla p_{\text{mag}}. \tag{9.82}$$

Because of the fact that $\delta\mathbf{B}\cdot\mathbf{v}_{\text{A}} \propto \delta\mathbf{B}\cdot\mathbf{B}_0 = 0$ for Alfvén waves we have for their contribution to the magnetic pressure

$$p_{\text{mag}} = \frac{|\delta \mathbf{B}|^2}{8\pi}. \tag{9.83}$$

However, since $\delta\mathbf{B}$ is sinusoidal in nature, a time-integrated values for the pressure due to Alfvén waves becomes

$$p_{\text{wave}} = \frac{|\delta \mathbf{B}|^2}{16\pi}. \tag{9.84}$$

It follows that during the collapse of a magnetized cloud, such as considered in Section 9.2, that Alfvén waves can bring support along any direction, including that of the general background magnetic field $\mathbf{B}_0$, and alleviate the problem brought about by the prediction of highly flatten clouds. This is because any decrease in the wave pressure (or in $|\delta\mathbf{B}|^2$, which is expected as one goes to regions of lower densities) in a given direction is accompanied by a force pointing in the same direction, as can be verified from equation (9.82).

Appendix – The Fast and Slow Modes, and Alfvén Waves

Without any loss of generality we can set $\mathbf{B}_0 = B_0\mathbf{e}_z$ and $\mathbf{k} = k_x\mathbf{e}_x + k_z\mathbf{e}_z$, then equation (9.77) can easily be put in a matrix form with

$$\begin{bmatrix} c_s^2 k_x^2 + v_A^2 k^2 - \omega^2 & 0 & c_s^2 k_x k_z \\ 0 & v_A^2 k_z^2 - \omega^2 & 0 \\ c_s^2 k_x k_z & 0 & c_s^2 k_z^2 - \omega^2 \end{bmatrix} \begin{pmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{pmatrix} = 0. \tag{9.85}$$

For this relation to have non-trivial solutions, the determinant of the matrix has to equal zero. A little bit of mathematical manipulation quickly yields

$$\left(\omega^2 - v_A^2 k_z^2\right)\left[\omega^4 - \left(c_s^2 + v_A^2\right)k^2\omega^2 + c_s^2 v_A^2 k^2 k_z^2\right] = 0. \tag{9.86}$$

This is a third order polynomial in ω^2, with the following roots

$$\omega^2 = \begin{cases} v_A^2 k^2 \cos^2(\theta) \\ \dfrac{k^2}{2}\left(c_s^2 + v_A^2\right) \pm \dfrac{k^2}{2}\left[\left(c_s^2 + v_A^2\right)^2 - 4c_s^2 v_A^2 \cos^2(\theta)\right]^{1/2} \end{cases}, \tag{9.87}$$

or

$$\frac{\omega}{k}=\begin{cases} v_{\mathrm{A}}k\cos(\theta) \\ \left\{\frac{1}{2}\left(c_{\mathrm{s}}^{2}+v_{\mathrm{A}}^{2}\right)\pm\frac{1}{2}\left[\left(c_{\mathrm{s}}^{2}+v_{\mathrm{A}}^{2}\right)^{2}-4c_{\mathrm{s}}^{2}v_{\mathrm{A}}^{2}\cos^{2}(\theta)\right]^{1/2}\right\}^{1/2} \end{cases}, \tag{9.88}$$

where we set $k_z = k\cos(\theta)$. The '$+$' is for the so-called **fast mode**, while the '$-$' for the **slow mode** of wave propagation.

Obtaining the eigenvectors implies solving for the following equation

$$\begin{bmatrix} c_{\mathrm{s}}^{2}k_{x}^{2}+v_{\mathrm{A}}^{2}k^{2} & 0 & c_{\mathrm{s}}^{2}k_{x}k_{z} \\ 0 & v_{\mathrm{A}}^{2}k_{z}^{2} & 0 \\ c_{\mathrm{s}}^{2}k_{x}k_{z} & 0 & c_{\mathrm{s}}^{2}k_{z}^{2} \end{bmatrix}\begin{pmatrix} \delta u_{x} \\ \delta u_{y} \\ \delta u_{z} \end{pmatrix}=\omega^{2}\begin{pmatrix} \delta u_{x} \\ \delta u_{y} \\ \delta u_{z} \end{pmatrix}. \tag{9.89}$$

It is clear from equation (9.89) that the first mode is for the Alvén waves discussed previously, which when $\omega^2 = v_{\mathrm{A}}^2 k_z^2$ has the eigenvector $\delta u_x = \delta u_z = 0$, and $\delta u_y = 1$. This mode is therefore transverse as $\delta\mathbf{u}\cdot\mathbf{v}_{\mathrm{A}} = \delta\mathbf{u}\cdot\mathbf{k} = 0$. Because of the required orthogonality between the different eigenvectors, the other two eigenvectors must have $\delta u_y = 0$. It follows that the first and third rows of equation (9.89) can be used to determine the δu_x and δu_z components, but since we only require the ratio of these components, one of the two equations will suffice. Using the first row, we find

$$\delta u_x : \delta u_z = \omega^2 - c_{\mathrm{s}}^2 k_x^2 - v_{\mathrm{A}}^2 k^2 : c_{\mathrm{s}}^2 k_x k_z. \tag{9.90}$$

Inserting the last of equations (9.87) into equation (9.90) we get

$$\begin{aligned}\delta u_x : \delta u_z = c_{\mathrm{s}}^2\left[1-2\sin^2(\theta)\right] - v_{\mathrm{A}}^2 \pm \left[\left(c_{\mathrm{s}}^2+v_{\mathrm{A}}^2\right)^2 - 4c_{\mathrm{s}}^2 v_{\mathrm{A}}^2\cos^2(\theta)\right]^{1/2} \\ : 2c_{\mathrm{s}}^2\sin(\theta)\cos(\theta).\end{aligned} \tag{9.91}$$

It can be verified that the two eigenvectors specified by equation (9.91) are orthogonal. These vectors both have velocity components parallel (longitudinal mode) and perpendicular (transverse mode) to $\mathbf{v}_{\mathrm{A}}$.

When $v_{\mathrm{A}} \gg c_{\mathrm{s}}$ (as is often the case in the turbulent regions of molecular clouds) the phase velocities for the fast and slow modes are

$$
\begin{aligned}
\frac{\omega}{k} &\simeq \frac{v_{\mathrm{A}}^2}{2}\left(1+\frac{c_{\mathrm{s}}^2}{v_{\mathrm{A}}^2} \pm \left\{1+2\frac{c_{\mathrm{s}}^2}{v_{\mathrm{A}}^2}\left[1-2\cos^2(\theta)\right]\right\}^{1/2}\right) \\
&\simeq \frac{v_{\mathrm{A}}^2}{2}\left(1+\frac{c_{\mathrm{s}}^2}{v_{\mathrm{A}}^2} \pm \left\{1+\frac{c_{\mathrm{s}}^2}{v_{\mathrm{A}}^2}\left[1-2\cos^2(\theta)\right]\right\}\right) \\
&\simeq \frac{v_{\mathrm{A}}^2}{2}\left(\left[1\pm 1\right]+\frac{c_{\mathrm{s}}^2}{v_{\mathrm{A}}^2}\left\{1\pm\left[1-2\cos^2(\theta)\right]\right\}\right).
\end{aligned}
\tag{9.92}
$$

And the corresponding eigenvectors are

$$
\begin{aligned}
\delta u_x : \delta u_z &\simeq \frac{c_{\mathrm{s}}^2}{v_{\mathrm{A}}^2}\left[1-2\sin^2(\theta)\right]-1\pm\left\{1+\frac{c_{\mathrm{s}}^2}{v_{\mathrm{A}}^2}\left[1-2\cos^2(\theta)\right]\right\} : 2\frac{c_{\mathrm{s}}^2}{v_{\mathrm{A}}^2}\sin(\theta)\cos(\theta). \\
&\simeq -1\pm 1+\frac{c_{\mathrm{s}}^2}{v_{\mathrm{A}}^2}\left[1\pm 1-2\sin^2(\theta)\mp 2\cos^2(\theta)\right] : 2\frac{c_{\mathrm{s}}^2}{v_{\mathrm{A}}^2}\sin(\theta)\cos(\theta).
\end{aligned}
\tag{9.93}
$$

For the fast mode we have

$$
\begin{aligned}
\left(\frac{\omega}{k}\right)^2 &\simeq v_{\mathrm{A}}^2\left[1+\frac{c_{\mathrm{s}}^2}{v_{\mathrm{A}}^2}\sin^2(\theta)\right] \\
\delta u_x : \delta u_z &\simeq 0 : 2\frac{c_{\mathrm{s}}^2}{v_{\mathrm{A}}^2}\sin(\theta)\cos(\theta),
\end{aligned}
\tag{9.94}
$$

and the only component of velocity oscillation is parallel to $\mathbf{v}_{\mathrm{A}}$ ($\mathbf{B}_0$). Moreover, the continuity and induction equations (see equations (9.73)) yield

$$
\begin{aligned}
\delta\rho &= \rho_0\frac{k_z}{\omega}\delta u_z \\
&\propto \sin(\theta)\cos(\theta) \\
\delta\mathbf{B} &= 0,
\end{aligned}
\tag{9.95}
$$

These equations indicate in this limit (i.e., $v_{\mathrm{A}} \gg c_{\mathrm{S}}$) that the fast mode generate compression in the gas (propagating at the Alfvén speed, approximately) that could cause them to dissipate, and that there are no perturbations in the background magnetic field. For the slow mode

$$\left(\frac{\omega}{k}\right)^2 \simeq c_{\mathrm{s}}^2 \cos^2(\theta)$$

$$\begin{aligned}\delta u_x : \delta u_z &\simeq -2\left\{1+\frac{c_{\mathrm{s}}^2}{v_{\mathrm{A}}^2}\left[\sin^2(\theta)-\cos^2(\theta)\right]\right\} : 2\frac{c_{\mathrm{s}}^2}{v_{\mathrm{A}}^2}\sin(\theta)\cos(\theta) \\ &\simeq 1:0,\end{aligned} \tag{9.96}$$

since $\delta u_x \gg \delta u_z$. That is, the oscillations happen in a direction perpendicular to $\mathbf{v}_{\mathrm{A}}$ and in the plane defined by $\mathbf{k}$ and $\mathbf{v}_{\mathrm{A}}$. The continuity and induction equations yield

$$\begin{aligned}\delta\rho &= \rho_0 \frac{k_x}{\omega}\delta u_x \\ \delta\mathbf{B} &= -\frac{\mathbf{k}}{\omega}\times\left(\delta u_x \mathbf{e}_x \times B_0 \mathbf{e}_z\right) \\ &= \frac{k}{\omega}B_0 \delta u_x\left[\sin(\theta)\mathbf{e}_z - \cos(\theta)\mathbf{e}_x\right],\end{aligned} \tag{9.97}$$

which indicate that in this limit (i.e., $v_{\mathrm{A}} \gg c_{\mathrm{S}}$) the slow mode also generates compression in the gas (propagating at the sound speed, approximately) that could bring dissipation, and that the perturbation in the background magnetic field are perpendicular to the direction of propagation (i.e., $\mathbf{k}$).